New approach to Schensted-Knuth normal forms*

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Abstract: We present the plactic algebra on an arbitrary alphabet set A by row generators and we give a Gröbner-Shirshov basis for such a presentation. From the Composition-Diamond lemma for associative algebras, it follows that the set of Young tableaux is a linear basis of plactic algebra. As the result, it gives a new proof that Young tableaux are normal forms of elements of plactic monoid. This result was proved by D.E. Knuth [6] in 1970, see also Chapter 5 in [8].

Key words: Gröbner-Shirshov basis, normal form, associative algebra, plactic monoid, plactic algebra, Young tableau.

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1 Introduction

Let $A = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$. Then we call

$$Pl(A) := sgp\langle A|\Omega\rangle = A^*/\equiv$$

a plactic monoid on the alphabet set A, see [8], where A^* is the free monoid generated by A, $\equiv$ is the congruence of A^* generated by the Knuth relations Ω and Ω consists of

$$\sigma_i\sigma_k\sigma_j = \sigma_k\sigma_i\sigma_j \ (i \leq j < k),$$
$$\sigma_j\sigma_i\sigma_k = \sigma_j\sigma_k\sigma_i \ (i < j \leq k).$$

Let F be a field. Then $F\langle A|\Omega\rangle$ is called the plactic monoid algebra over F of $Pl(A)$.

The basic theory of plactic monoid was systematically developed in Lascoux and Sch$\ddot{u}$tzenberger [9] in 1981.

D.E. Knuth [6] in 1970, see also Chapter 5 in [8], shows that Young tableaux are normal forms of elements of plactic monoid.

In a recent paper [7], with the deg-lex ordering on A^*, a finite Gröbner-Shirshov basis for plactic algebra is given when $n = 3$. They prove that for $n > 3$, a corresponding Gröbner-Shirshov basis must be infinite. In fact, it is an open problem to give a corresponding Gröbner-Shirshov basis by using such a presentation and the deg-lex ordering on A^*.

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In this paper, we present the plactic algebra on an arbitrary alphabet set A by row generators. We give a Gröbner-Shirshov basis for such the presentation. From the Composition-Diamond lemma for associative algebras, it follows that the set of Young tableaux is a linear basis of plactic algebra. As the result, it gives a new proof that Young tableaux are normal forms of elements of plactic monoid.

2 Preliminaries

We first cite some concepts and results from the literatures [2, 3, 11, 12] which are related to Gröbner-Shirshov bases for associative algebras.

Let X be a set and F a field. Throughout this paper, we denote $F\langle X\rangle$ the free associative algebra over F generated by X, X^* the free monoid generated by X and N the set of non-negative integers.

A well ordering $<$ on X^* is called monomial if for $u, v \in X^*$, we have

$$u < v \Rightarrow w|_u < w|_v \ \ for\ all\ w \in X^*,$$

where $w|_u = w|_{x_i \mapsto u},\ w|_v = w|_{x_i \mapsto v}$ and x_i's are the same individuality of the letter $x_i \in X$ in w.

A standard example of monomial ordering on X^* is the deg-lex ordering which first compare two words by degree and then by comparing them lexicographically, where X is a well-ordered set.

Let X^* be a set with a monomial ordering $<$. Then, for any non-zero polynomial $f \in F\langle X\rangle$, f has the leading word $\overline{f}$. We call f monic if the coefficient of $\overline{f}$ is 1. By $|\overline{f}|$ we denote the degree (length) of $\overline{f}$.

Let $f,\ g \in F\langle X\rangle$ be two monic polynomials and $w \in X^*$. If $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ such that $|\overline{f}| + |\overline{g}| > |w|$, then $(f, g)_w = fb - ag$ is called the intersection composition of f, g relative to w. If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then $(f, g)_w = f - agb$ is called the inclusion composition of f, g relative to w.

Let $S \subset F\langle X\rangle$ be a monic set. A composition $(f, g)_w$ is called trivial modulo (S, w), denoted by

$$(f, g)_w \equiv 0 \quad mod(S, w)$$

if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where every $\alpha_i \in k,\ s_i \in S,\ a_i, b_i \in X^*$, and $a_i\overline{s_i}b_i < w$.

Recall that S is a Gröbner-Shirshov basis in $F\langle X\rangle$ if any composition of polynomials from S is trivial modulo S and corresponding w.

The following lemma was first proved by Shirshov [11, 12] for free Lie algebras (with deg-lex ordering) (see also Bokut [2]). Bokut [3] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For commutative polynomials, this lemma is known as Buchberger's Theorem (see [4, 5]).

Lemma 2.1 *(Composition-Diamond lemma for associative algebras) Let F be a field, $<$ a monomial ordering on X^* and $Id(S)$ the ideal of $F\langle X\rangle$ generated by S. Then the following statements are equivalent:*

1) S is a Gröbner-Shirshov basis in $F\langle X\rangle$.

2) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ *for some* $s \in S$ *and* $a, b \in X^*$.

3) $Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ *is an* F*-basis of the algebra* $A = F\langle X|S\rangle = F\langle X\rangle/Id(S)$.

If a subset S of $F\langle X\rangle$ is not a Gröbner-Shirshov basis then one can add all nontrivial compositions of polynomials of S to S. Continuing this process repeatedly, we finally obtain a Gröbner-Shirshov basis S^{comp} that contains S. Such a process is called Shirshov algorithm.

Let $A = sgp\langle X|S\rangle$ be a semigroup representation. Then S is also a subset of $F\langle X\rangle$ and we can find Gröbner-Shirshov basis S^{comp}. We also call S^{comp} a Gröbner-Shirshov basis of A. $Irr(S^{comp}) = \{u \in X^* | u \neq a\overline{f}b,\ a, b \in X^*,\ f \in S^{comp}\}$ is an F-basis of $F\langle X|S\rangle$ which is also a normal form of the semigroup A.

3 Main theorem

Let $A = \{\sigma_i | i \in I\}$ be a well-ordered alphabet set. Then we call

$$Pl(A) := sgp\langle A|\Omega\rangle = A^*/\equiv$$

a plactic monoid on the alphabet set A, see [8], where $\equiv$ is the congruence of A^* generated by the Knuth relations Ω and Ω consists of

$$\sigma_i\sigma_k\sigma_j = \sigma_k\sigma_i\sigma_j \ (\sigma_i \leq \sigma_j < \sigma_k),$$
$$\sigma_j\sigma_i\sigma_k = \sigma_j\sigma_k\sigma_i \ (\sigma_i < \sigma_j \leq \sigma_k).$$

Now, suppose that $A = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$.

Let N be the set of non-negative integers. A word $R \in A^*$ is called a row if it is nondecreasing, for example, $R = 111225$ is a row. For convenience, denote $R = (r_1, r_2, \ldots, r_n)$, where r_i is the number of letter i $(i = 1, 2, \ldots, n)$, for example, $R = 111225 = (3, 2, 0, 0, 1, 0, \ldots, 0)$.

Let $U = \{R \in A^* \mid R \text{ is a row}\}$.

We order the set U^* as follows.

Let $R = (r_1, r_2, \ldots, r_n) \in U$. Then $|R| = r_1 + \cdots + r_n$ is the length of R in A^*.

We first order U: for any $R, S \in U$, $R < S$ if and only if $|R| < |S|$ or $|R| = |S|$ and there exists a t $(0 \leq t < n)$ such that $r_i = s_i,\ i = 1, \ldots, t$ and $r_{t+1} > s_{t+1}$. Clearly, this is a well ordering on U.

Then, we order U^* by the deg-lex ordering. We will use this ordering throughout this paper.

Let $R, S \in U$. Then R dominates S if $|R| \leq |S|$ and each letter of R is larger than the corresponding letter of S.

A (semistandard) Young tableau on A (see [8]) is a word $w = R_1R_2 \cdots R_t \in U^*$ such that R_i dominates $R_{i+1},\ i = 1, \ldots, t-1$. For example, $4556 \cdot 223357 \cdot 1112444$ is a Young tableau.

The following algorithm was introduced by Schensted [10] in 1961.

Definition 3.1 *(Schensted's algorithm [10])*
Let $R \in U$, $x \in A$.

$$R \cdot x = \begin{cases} Rx, & if\ Rx\ is\ a\ row, \\ y \cdot R', & otherwise \end{cases}$$

where y is the leftmost letter in R and is strictly larger than x, and $R' = R\mid_{y \mapsto x}$, i.e., R' is obtained from R by replacing y by x.

Then, for any $R, S \in U$, by induction, it is clear that there exist uniquely $R', S' \in U$ such that $R \cdot S = R' \cdot S'$ and $R' \cdot S'$ is a Young tableau, where R' is empty if $R \cdot S = S'$ is a row.

It is clear that

$$sgp\langle U \mid \Lambda\rangle = sgp\langle A \mid \Omega\rangle$$

and so $F\langle U \mid \Lambda\rangle = F\langle A \mid \Omega\rangle$, where $\Lambda = \{R \cdot S = R' \cdot S',\ R, S \in U\}$.

The following is the main theorem of the paper.

Theorem 3.2 *Let the notation be as before. Then with the deg-lex ordering on U^*, Λ is a Gröbner-Shirshov basis for the plactic algebra $F\langle U \mid \Lambda\rangle$.*

By using Composition-Diamond lemma for associative algebras (Lemma 2.1) and Theorem 3.2, we have the following corollary.

Corollary 3.3 *([8], Chapter 5) The set of Young tableaux on A is a normal form of the plactic monoid $sgp\langle A \mid \Omega\rangle$.*

We will prove Theorem 3.2 by assuming that $A = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$.

Remark: For an arbitrary well-ordered set A, we define similarly a row on A^*, Schensted's algorithm, Young tableau on A and the set Λ. Then for an arbitrary well-ordered set A, similar to the proof of A to be finite, we also have Theorem 3.2 and Corollary 3.3.

4 A proof of Theorem 3.2

4.1 Technique formulas

Let $(\phi_1, \ldots, \phi_n) \in U$. Denote $\Phi_p = \sum_{i=1}^{p} \phi_i$ $(1 \le p \le n)$, where ϕ represents any lowercase symbol and Φ the corresponding uppercase symbol.

Definition 4.1 *Let $W = (w_1, w_2, \ldots, w_n), Z = (z_1, z_2, \ldots, z_n) \in U$. Define an algorithm*

$$\begin{aligned} W \cdot Z &= (w_1, w_2, \ldots, w_n)(z_1, z_2, \ldots, z_n) \\ &= (x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) \\ &= X \cdot Y \end{aligned}$$

where $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n),\ x_1 = 0$ *and*

$$\begin{aligned} x_p &= min(Z_{p-1} - X_{p-1}, w_p), \\ X_p &= min(Z_{p-1}, X_{p-1} + w_p), \\ y_q &= w_q + z_q - x_q, \\ Y_q &= W_q + Z_q - X_q, \end{aligned}$$

where $n \geq p \geq 2,\ n \geq q \geq 1$. *Clearly, either* $X, Y \in U$ *or* $Y \in U$ *and* $X = (0, 0, \ldots, 0)$.

The formulas in Definition 4.1 are very useful in the proof of Theorem 3.2.

Lemma 4.2 *The algorithms in Definition 4.1 and Definition 3.1 are equivalent.*

Proof. Definition 4.1 $\Rightarrow$ Definition 3.1.

Suppose that $W = (w_1, w_2, \ldots, w_n) \in U$ and Z is a letter. Then we can express $Z = (0, \ldots, 0, z_p, 0, \ldots, 0)$, where $z_p = 1$ $(1 \leq p \leq n)$.

Clearly $Z_1 = Z_2 = \cdots = Z_{p-1} = 0$ and $Z_p = Z_{p+1} = \cdots = Z_n = 1$. Since $x_1 = X_1 = 0$ and Definition 4.1, $x_2 = min(Z_1 - X_1, w_2) = 0$. Similarly, $x_3 = x_4 = \cdots = x_p = 0$.

Let w_j $(1 \leq j \leq n)$ satisfy $w_j \neq 0$ and $w_{j+1} = \cdots = w_n = 0$. There are two cases to consider.

Case 1. $p \geq j$.

By Definition 4.1, $x_{p+1} = min(Z_p - X_p, w_{p+1}) = min(1 - 0, 0) = 0$. Similarly, $x_{j+1} = \cdots = x_n = 0$, i.e., $X = (0, \ldots, 0)$. Then $Y = (w_1, \ldots, w_{p-1}, w_p + 1, w_{p+1}, \ldots, w_n)$. This means $WZ = Y$ is a row which corresponds to the first part of Definition 3.1.

Case 2. $p < j$.

There must exist a $w_t, p < t \leq j$ such that $w_t \neq 0$ and $w_k = 0,\ p < k < t$. If $p + 1 = t$, then $x_t = min(Z_{t-1} - X_{t-1}, w_t) = min(1 - 0, w_t) = 1$, and $x_i = min(Z_{i-1} - X_{i-1}, w_i) = min(1 - 1, w_i) = 0, t + 1 \leq i \leq n$. If $p + 1 < t$, then by Definition 4.1, $x_{p+1} = min(Z_p - X_p, w_{p+1}) = min(1 - 0, 0) = 0$. Similarly, $x_{p+2} = \cdots = x_{t-1} = 0$. Moreover $x_t = min(Z_{t-1} - X_{t-1}, w_t) = min(1 - 0, w_t) = 1$, $x_{t+1} = min(Z_t - X_t, w_{t+1}) = 0$. Similarly, $x_{t+2} = \cdots = x_n = 0$. This shows that $X = (0, \ldots, 0, x_t, 0, \ldots, 0)$, where $x_t = 1$. Now, $y_p = w_p + z_p - x_p = w_p + 1$, $y_t = w_t + z_t - x_t = w_t - 1$ and $y_i = w_i + z_i - x_i = w_i, 1 \leq i \leq n, i \neq p, i \neq t$, i.e., $Y = (w_1, \ldots, w_p + 1, \ldots, w_t - 1, \ldots, w_n)$, which corresponds to the second part of Definition 3.1.

Definition 3.1 $\Rightarrow$ Definition 4.1.

Clearly, $x_1 = 0$ and $x_2 = min(z_1, w_2) = min(Z_1 - X_1, w_2)$.

Assume that for all $t \leq m - 1$ $(t \geq 1,\ m \geq 3)$, $x_t = min(Z_{t-1} - X_{t-1}, w_t)$. Then

$$\begin{aligned} x_m &= min(Z_{m-2} - X_{m-2} - x_{m-1} + z_{m-1}, w_m) \\ &= min(Z_{m-1} - X_{m-1}, w_m) \end{aligned}$$

which shows that $x_p = min(Z_{p-1} - X_{p-1}, w_p)$ for any p.

It is obvious that $X_p = x_p + X_{p-1} = min(Z_{p-1}, X_{p-1} + w_p),\ y_q = w_q + z_q - x_q$ and $Y_q = W_q + Z_q - X_q$. $\square$

Corollary 4.3 *In Definition 4.1,* $X \cdot Y$ *is a Young tableau.*

4.2 Expressions of reductions

In order to prove the Theorem 3.2, we have to check that all possible compositions in Λ, which are only intersections, are trivial.

Let $R = (r_1, r_2, \ldots, r_n),\ S = (s_1, s_2, \ldots, s_n),\ T = (t_1, t_2, \ldots, t_n) \in U$.

For reductions, we will use the following notation.

$$\begin{aligned} RST \quad &= \begin{pmatrix} r_1 & r_2 & r_3 & \ldots & r_n \\ s_1 & s_2 & s_3 & \ldots & s_n \\ t_1 & t_2 & t_3 & \ldots & t_n \end{pmatrix} \overset{S\cdot T = A\cdot B}{\Longrightarrow} \begin{pmatrix} r_1 & r_2 & r_3 & \ldots & r_n \\ a_1 & a_2 & a_3 & \ldots & a_n \\ b_1 & b_2 & b_3 & \ldots & b_n \end{pmatrix} \\ &\overset{R\cdot A = C\cdot D}{\Longrightarrow} \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \\ d_1 & d_2 & d_3 & \ldots & d_n \\ b_1 & b_2 & b_3 & \ldots & b_n \end{pmatrix} \overset{D\cdot B = E\cdot F}{\Longrightarrow} \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \\ e_1 & e_2 & e_3 & \ldots & e_n \\ f_1 & f_2 & f_3 & \ldots & f_n \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} RST \quad &= \begin{pmatrix} r_1 & r_2 & r_3 & \ldots & r_n \\ s_1 & s_2 & s_3 & \ldots & s_n \\ t_1 & t_2 & t_3 & \ldots & t_n \end{pmatrix} \overset{R\cdot S = G\cdot H}{\Longrightarrow} \begin{pmatrix} g_1 & g_2 & g_3 & \ldots & g_n \\ h_1 & h_2 & h_3 & \ldots & h_n \\ t_1 & t_2 & t_3 & \ldots & t_n \end{pmatrix} \\ &\overset{H\cdot T = I\cdot J}{\Longrightarrow} \begin{pmatrix} g_1 & g_2 & g_3 & \ldots & g_n \\ i_1 & i_2 & i_3 & \ldots & i_n \\ j_1 & j_2 & j_3 & \ldots & j_n \end{pmatrix} \overset{G\cdot I = K\cdot L}{\Longrightarrow} \begin{pmatrix} k_1 & k_2 & k_3 & \ldots & k_n \\ l_1 & l_2 & l_3 & \ldots & l_n \\ j_1 & j_2 & j_3 & \ldots & j_n \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} &A = (a_1, a_2, \ldots, a_n), \quad B = (b_1, b_2, \ldots, b_n), \quad C = (c_1, c_2, \ldots, c_n), \\ &D = (d_1, d_2, \ldots, d_n), \quad E = (e_1, e_2, \ldots, e_n), \quad F = (f_1, f_2, \ldots, f_n), \\ &G = (g_1, g_2, \ldots, g_n), \quad H = (h_1, h_2, \ldots, h_n), \quad I = (i_1, i_2, \ldots, i_n), \\ &J = (j_1, j_2, \ldots, j_n), \quad K = (k_1, k_2, \ldots, k_n), \quad L = (l_1, l_2, \ldots, l_n), \end{aligned}$$

and $S\cdot T = A\cdot B, R\cdot A = C\cdot D, D\cdot B = E\cdot F, R\cdot S = G\cdot H, H\cdot T = I\cdot J, G\cdot I = K\cdot L \in \Lambda$.

We will prove that

$$\begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \\ e_1 & e_2 & e_3 & \ldots & e_n \\ f_1 & f_2 & f_3 & \ldots & f_n \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & k_3 & \ldots & k_n \\ l_1 & l_2 & l_3 & \ldots & l_n \\ j_1 & j_2 & j_3 & \ldots & j_n \end{pmatrix},$$

which implies that the intersection composition $(RS, ST)_{RST} \equiv 0, mod(\Lambda, RST)$. Therefore, Λ is a Gröbner-Shirshov basis of the algebra $F\langle U \mid \Lambda\rangle$.

By the definition of the algorithm in Definition 4.1, $a_1 = c_1 = c_2 = e_1 = g_1 = i_1 = k_1 = k_2 = l_1 = 0$. Therefore, one needs only to show that $c_p = k_p$ and $e_q = l_q$ for all $3 \le p \le n, 2 \le q \le n$.

4.3 $c_p = k_p$ $(p \ge 3)$

We need the following lemmas to prove $c_p = k_p$ $(p \ge 3)$.

Lemma 4.4 *For all* p $(1 \leq p \leq n)$, $A_p \leq I_p$.

Proof. By the algorithm in Definition 4.1, $A_1 = a_1 = i_1 = I_1 = 0$. Assume that for all $p \leq m-1$ $(m \geq 2)$, $A_p \leq I_p$. Since

$$\begin{aligned} A_m &= min(T_{m-1}, A_{m-1} + s_m), \\ I_m &= min(T_{m-1}, I_{m-1} + h_m), \\ h_m &= r_m + s_m - g_m \\ &= r_m + s_m - min(S_{m-1} - G_{m-1}, r_m) \geq s_m, \end{aligned}$$

we have $A_m \leq I_m$. Now by induction, the result follows. □

Lemma 4.5 *Assume that* $p \geq 3$, $K_{p-1} = C_{p-1}$, $K_p = C_p = A_{p-1} < C_{p-1} + r_p$ *and* $S_{p-1} - G_{p-1} \geq r_p$. *Then* $K_p = I_{p-1}$.

Proof. Note that $C_p = min(A_{p-1}, C_{p-1} + r_p)$ and $K_p = min(I_{p-1}, K_{p-1} + g_p)$.

If $S_{p-1} - G_{p-1} \geq r_p$, then $g_p = min(S_{p-1} - G_{p-1}, r_p) = r_p$.

Therefore $K_p = C_p = A_{p-1} < C_{p-1} + r_p = K_{p-1} + g_p$, which concludes $K_p = I_{p-1}$. □

Lemma 4.6 *Assume that* $p \geq 2$, $C_p = K_p$, $C_{p+1} = K_{p+1}$ *and* $C_{p+1} = C_p + r_{p+1}$. *Then* $S_p - G_p \geq r_{p+1}$.

Proof. Note that $C_{p+1} = min(A_p, C_p + r_{p+1})$ and $K_{p+1} = min(I_p, K_p + g_p) = min(I_p, K_p + S_p - G_p, K_p + r_{p+1})$.

If $C_p = K_p, C_{p+1} = K_{p+1}$ and $C_{p+1} = C_p + r_{p+1}$, then $K_{p+1} = C_{p+1} = C_p + r_{p+1} = K_p + r_{p+1}$. Therefore, $I_p \geq K_p + r_{p+1}$, $K_p + S_p - G_p \geq K_p + r_{p+1}$, which concludes $S_p - G_p \geq r_{p+1}$. □

4.3.1 $C_3 = K_3$

Proof. Since $c_1 = c_2 = k_1 = k_2 = 0$, we have $C_3 = c_3$ and $K_3 = k_3$. According to the algorithm in Definition 4.1,

$$\begin{aligned} c_3 &= min(a_2, r_3) \\ &= min[min(t_1, s_2), r_3] \\ &= min(t_1, s_2, r_3), \\ k_3 &= min(i_2, g_3) \\ &= min[min(t_1, h_2), min(s_1 + s_2 - g_2, r_3)] \\ &= min[t_1, r_2 + s_2 - min(s_1, r_2), s_1 + s_2 - min(s_1, r_2), r_3] \\ &= min[t_1, s_2 + min(r_2, s_1) - min(s_1, r_2), r_3] \\ &= min(t_1, s_2, r_3). \end{aligned}$$

Therefore, $C_3 = K_3 = min(t_1, s_2, r_3)$. □

4.3.2 $C_p = K_p$ $(p \geq 4)$

Proof. Assume that for all $p \leq m-1$ $(m \geq 4)$, $C_p = K_p$. Note that

$$\begin{aligned} C_m &= min(T_{m-2}, A_{m-2} + s_{m-1}, C_{m-1} + r_m), \\ K_m &= min(T_{m-2}, I_{m-2} + h_{m-1}, K_{m-1} + M, K_{m-1} + r_m), \end{aligned}$$

where $M = s_{m-1} + S_{m-2} - G_{m-2} - g_{m-1}$.

There are three cases to consider.

Case 1. $C_{m-1} = A_{m-2}$.

If $A_{m-2} < C_{m-2} + r_{m-1}$, $S_{m-2} - G_{m-2} < r_{m-1}$, then

$$\begin{aligned} &C_{m-1} = A_{m-2}, \\ &g_{m-1} = S_{m-2} - G_{m-2}, \\ &h_{m-1} = r_{m-1} + s_{m-1} - S_{m-2} + G_{m-2} > s_{m-1}, \\ &M = s_{m-1}, \\ &I_{m-2} + h_{m-1} \geq K_{m-1} + h_{m-1} > K_{m-1} + s_{m-1}. \end{aligned}$$

Therefore, $K_m = C_m = min(T_{m-2}, C_{m-1} + s_{m-1}, C_{m-1} + r_m)$.

If $A_{m-2} < C_{m-2} + r_{m-1}$, $S_{m-2} - G_{m-2} \geq r_{m-1}$, then $C_{m-1} = A_{m-2}$, $g_{m-1} = r_{m-1}$, $h_{m-1} = s_{m-1}$ and $M \geq s_{m-1}$. By Lemma 4.5, we have $K_{m-1} = I_{m-2}$ and $I_{m-2} + h_{m-1} = K_{m-1} + s_{m-1}$. Therefore, $K_m = C_m = min(T_{m-2}, C_{m-1} + s_{m-1}, C_{m-1} + r_m)$.

Case 2. $C_{m-1} = A_{m-q-2} + \sum_{i=1}^{q+1} r_{m-i}$, where $q \leq m-4$.

The condition $C_{m-1} = A_{m-q-2} + \sum_{i=1}^{q+1} r_{m-i}$ implies that for all $k \leq q$ $(q \leq m-4)$, $C_{m-k-1} + r_{m-k} \leq A_{m-k-1}$. Then $C_{m-k} = C_{m-k-1} + r_{m-k}$ and by Lemma 4.6, we have

$$\begin{aligned} &S_{m-k-1} - G_{m-k-1} \geq r_{m-k}, \\ &g_{m-k} = r_{m-k}, \\ &h_{m-k} = s_{m-k}, \end{aligned}$$

$$\begin{aligned} C_m &= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_{m-q-2} + \sum_{i=1}^{q} s_{m-i}, A_{m-q-2} + \sum_{i=1}^{q+1} s_{m-i}, C_{m-1} + r_m), \\ K_m &= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_{m-q-2} + \sum_{i=1}^{q} s_{m-i}, \\ &\qquad I_{m-q-2} + h_{m-q-1} + \sum_{i=1}^{q} s_{m-i}, K_{m-q-1} + \sum_{i=1}^{q} r_{m-i} + M, K_{m-1} + r_m), \end{aligned}$$

where $M = \sum_{i=1}^{q+1} s_{m-i} + S_{m-q-2} - G_{m-q-2} - g_{m-q-1} - \sum_{i=1}^{q} r_{m-i}$.

If $A_{m-q-2} < C_{m-q-2} + r_{m-q-1}$ and $S_{m-q-2} - G_{m-q-2} < r_{m-q-1}$, then

$$C_{m-q-1} = A_{m-q-2},$$
$$g_{m-q-1} = S_{m-q-2} - G_{m-q-2},$$
$$h_{m-q-1} = r_{m-q-1} + s_{m-q-1} - S_{m-q-2} + G_{m-q-2} > s_{m-q-1},$$
$$M = \sum_{i=1}^{q+1} s_{m-i} - \sum_{i=1}^{q} r_{m-i},$$
$$K_{m-q-1} + \sum_{i=1}^{q} r_{m-i} + M = K_{m-q-1} + \sum_{i=1}^{q+1} s_{m-i},$$
$$I_{m-q-2} + h_{m-q-1} + \sum_{i=1}^{q} s_{m-i} > K_{m-q-1} + \sum_{i=1}^{q+1} s_{m-i}.$$

Therefore,

$$\begin{aligned} K_m &= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_{m-q-2} + \sum_{i=1}^{q} s_{m-i}, C_{m-q-1} + \sum_{i=1}^{q+1} s_{m-i}, C_{m-1} + r_m) \\ &= C_m. \end{aligned}$$

If $A_{m-q-2} < C_{m-q-2} + r_{m-q-1}$ and $S_{m-q-2} - G_{m-q-2} \geq r_{m-q-1}$, then $C_{m-q-1} = A_{m-q-2}$, $g_{m-q-1} = r_{m-q-1}$ and $h_{m-q-1} = s_{m-q-1}$. By Lemma 4.5,

$$K_{m-q-1} = I_{m-q-2},$$
$$M \geq \sum_{i=1}^{q+1} s_{m-i} - \sum_{i=1}^{q} r_{m-i},$$
$$K_{m-q-1} + \sum_{i=1}^{q} r_{m-i} + M \geq K_{m-q-1} + \sum_{i=1}^{q+1} s_{m-i},$$
$$I_{m-q-2} + h_{m-q-1} + \sum_{i=1}^{q} s_{m-i} = K_{m-q-1} + \sum_{i=1}^{q+1} s_{m-i}.$$

Therefore,

$$K_m = min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_{m-q-2} + \sum_{i=1}^{q} s_{m-i}, C_{m-q-1} + \sum_{i=1}^{q+1} s_{m-i}, C_{m-1} + r_m) = C_m.$$

Case 3. $C_{m-1} = C_2 + \sum_{i=1}^{m-3} r_{m-i}$.

The condition $C_{m-1} = C_2 + \sum_{i=1}^{m-3} r_{m-i}$ implies that for all $k \leq q = m-3$, $C_{m-k-1} +$

$r_{m-k} \leq A_{m-k-1}$ and so $C_{m-k} = C_{m-k-1} + r_{m-k}$. Now, by Lemma 4.6,

$$
\begin{aligned}
& S_{m-k-1} - G_{m-k-1} \geq r_{m-k}, \\
& g_{m-k} = r_{m-k}, \\
& h_{m-k} = s_{m-k}, \\
M &= \sum_{i=1}^{m-2} s_{m-i} + S_1 - G_1 - g_2 - \sum_{i=1}^{m-3} r_{m-i} \\
&= \sum_{i=1}^{m-2} s_{m-i} + s_1 - g_2 - \sum_{i=1}^{m-3} r_{m-i}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
C_m &= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_1 + \sum_{i=1}^{m-3} s_{m-i}, A_1 + \sum_{i=1}^{m-2} s_{m-i}, C_{m-1} + r_m) \\
&= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_1 + \sum_{i=1}^{m-3} s_{m-i}, \sum_{i=1}^{m-2} s_{m-i}, C_{m-1} + r_m),
\end{aligned}
$$

and

$$
\begin{aligned}
K_m &= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_1 + \sum_{i=1}^{m-3} s_{m-i}, I_1 + h_2 + \sum_{i=1}^{m-3} s_{m-i}, \\
&\qquad K_2 + \sum_{i=1}^{m-3} r_{m-i} + M, K_{m-1} + r_m) \\
&= min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_1 + \sum_{i=1}^{m-3} s_{m-i}, \\
&\qquad \sum_{i=1}^{m-2} s_{m-i} + min(h_2 - s_2, s_1 - g_2), K_{m-1} + r_m).
\end{aligned}
$$

Since

$$
\begin{aligned}
min(h_2 - s_2, s_1 - g_2) &= min[r_2 - min(s_1, r_2), s_1 - min(s_1, r_2)] \\
&= min(r_2, s_1) - min(s_1, r_2) \\
&= 0,
\end{aligned}
$$

one gets

$$
K_m = min(T_{m-2}, T_{m-3} + s_{m-1}, \ldots, T_1 + \sum_{i=1}^{m-3} s_{m-i}, \sum_{i=1}^{m-2} s_{m-i}, K_{m-1} + r_m) = C_m.
$$

Therefore, the proof of $K_m = C_m$ is complete. $\square$

Therefore, we have shown that $C_p = K_p$ $(1 \leq p \leq n)$ which, by using the formulas in Definition 4.1, is clearly equivalent to $c_p = k_p$ $(1 \leq p \leq n)$.

4.4 $e_p = l_p$ $(p \geq 2)$

We need the following lemmas to prove $e_p = l_p$ $(p \geq 2)$.

Lemma 4.7 *For any p $(p \leq n)$, $S_p + C_p - G_p - A_p \geq 0$.*

Proof. For $p = 1$, we have $S_1 + C_1 - G_1 - A_1 = s_1 \geq 0$.
Assume that for all $p \leq m-1$ $(m \geq 2)$, $S_p + C_p - G_p - A_p \geq 0$.
If $A_{m-1} < C_{m-1} + r_m$, then $C_m = A_{m-1}$ and

$$S_m + C_m - G_m - A_m \geq S_m + A_{m-1} - S_{m-1} - A_{m-1} - s_m = 0.$$

If $C_{m-1} + r_m \leq A_{m-1}$, then $C_m = C_{m-1} + r_m$. By section 4.3, we have $K_m = C_m$ and $K_{m-1} = C_{m-1}$. Then by Lemma 4.6,

$$S_{m-1} - G_{m-1} \geq r_m,$$

$$G_m = G_{m-1} + r_m.$$

So,

$$\begin{aligned} S_m + C_m - G_m - A_m &= S_m + C_{m-1} + r_m - G_{m-1} - r_m - A_m \\ &= S_m + C_{m-1} - G_{m-1} - A_m \\ &= S_{m-1} + C_{m-1} - G_{m-1} - A_{m-1} + s_m - a_m \\ &\geq s_m - a_m \\ &= s_m - min(T_{m-1} - A_{m-1}, s_m) \\ &\geq 0. \end{aligned}$$

The proof is complete. □

Lemma 4.8 *If $S_p + C_p - G_p - A_p > 0$, then $A_p = I_p$.*

Proof. For $p = 1$, we have $A_1 = I_1 = 0$.
Assume that for all $p \leq m-1$ $(m \geq 2)$, $A_p = I_p$ if $S_p + C_p - G_p - A_p > 0$. By using Lemma 4.7, $S_m + C_m - G_m - A_m \geq 0$. Suppose that $S_m + C_m - G_m - A_m > 0$. Then by the proof of Lemma 4.7, one of the following four conditions should be satisfied.

(4.8.1) $A_{m-1} < C_{m-1} + r_m$, $A_m = T_{m-1} < A_{m-1} + s_m$;
(4.8.2) $A_{m-1} < C_{m-1} + r_m$, $G_m = G_{m-1} + r_m < S_{m-1}$;
(4.8.3) $C_{m-1} + r_m \leq A_{m-1}$, $S_{m-1} + C_{m-1} - G_{m-1} - A_{m-1} > 0$;
(4.8.4) $C_{m-1} + r_m \leq A_{m-1}$, $S_{m-1} + C_{m-1} - G_{m-1} - A_{m-1} = 0$, $s_m > a_m$.

Assume (4.8.1). Then $C_m = A_{m-1}$, and by Lemma 4.4, $A_m = T_{m-1} < A_{m-1} + s_m \leq I_{m-1} + h_m$. So, $A_m = I_m = T_{m-1}$.

Assume (4.8.2). Then $C_m = A_{m-1}$, $g_m = r_m$, $h_m = s_m$, and by Lemma 4.5, $K_m = I_{m-1}$. Thus, $A_m = I_m = min(T_{m-1}, C_m + s_m)$.

Assume (4.8.3). Then $C_m = C_{m-1} + r_m$. Since $K_m = C_m$ and $K_{m-1} = C_{m-1}$, by Lemma 4.6, we have $S_{m-1} - G_{m-1} \geq r_m$, $g_m = r_m$, $h_m = s_m$ and $A_{m-1} = I_{m-1}$. So, $A_m = I_m = min(T_{m-1}, A_{m-1} + s_m)$.

Assume (4.8.4). Then $C_m = C_{m-1} + r_m$ and $a_m = T_{m-1} - A_{m-1}$. By Lemma 4.4, $A_m = T_{m-1} < A_{m-1} + s_m \leq I_{m-1} + h_m$. So, $A_m = I_m = T_{m-1}$. □

4.4.1 $e_2 = l_2$

Proof.

$$\begin{aligned}
e_2 &= min(b_1, d_2) \\
&= min(s_1 + t_1, r_2 + a_2) \\
&= min[s_1 + t_1, r_2 + min(t_1, s_2)] \\
&= min(s_1 + t_1, t_1 + r_2, r_2 + s_2), \\
l_2 &= g_2 + i_2 \\
&= min(s_1, r_2) + min(t_1, h_2) \\
&= min(s_1, r_2) + min[t_1, r_2 + s_2 - min(s_1, r_2)] \\
&= min(s_1 + t_1, t_1 + r_2, r_2 + s_2) \\
&= e_2.
\end{aligned}$$

□

4.4.2 $e_p = l_p$ $(p \geq 3)$

Proof. Assume that for any $p \leq m - 1$ $(m \geq 3)$, $e_p = l_p$.

Since $E_{m-1} = L_{m-1} = G_{m-1} + I_{m-1} - K_{m-1}$ and $K_{m-1} = C_{m-1} = C_m - c_m$, we have

$$\begin{aligned}
e_m &= min(B_{m-1} - E_{m-1}, d_m) \\
&= min(S_{m-1} + T_{m-1} - A_{m-1} - E_{m-1}, r_m + a_m - c_m) \\
&= min[S_{m-1} + T_{m-1} - A_{m-1} - E_{m-1}, r_m + min(T_{m-1} - A_{m-1}, s_m) - c_m] \\
&= min(S_{m-1} + T_{m-1} + C_m - A_{m-1} - G_{m-1} - I_{m-1}, r_m + T_{m-1} - A_{m-1}, \\
&\quad r_m + s_m) - c_m, \\
l_m &= g_m + i_m - k_m \\
&= min(S_{m-1} - G_{m-1}, r_m) + min(T_{m-1} - I_{m-1}, h_m) - k_m \\
&= min(S_{m-1} + T_{m-1} - G_{m-1} - I_{m-1}, S_{m-1} - G_{m-1} + h_m, \\
&\quad r_m + T_{m-1} - I_{m-1}, r_m + h_m) - k_m.
\end{aligned}$$

There are two cases to consider.

Case 1. $A_{m-1} < C_{m-1} + r_m$.

Then $C_m = A_{m-1}$.

If $S_{m-1} - G_{m-1} < r_m$, then

$$\begin{aligned}
&g_m = S_{m-1} - G_{m-1}, \\
&h_m = r_m + s_m - S_{m-1} + G_{m-1} > s_m, \\
&S_{m-1} - G_{m-1} + h_m = r_m + s_m < r_m + h_m
\end{aligned}$$

and by Lemma 4.4,

$$\begin{aligned}
S_{m-1} + T_{m-1} - G_{m-1} - I_{m-1} &< r_m + T_{m-1} - I_{m-1} \\
&\leq r_m + T_{m-1} - A_{m-1}.
\end{aligned}$$

Therefore, $l_m = e_m = min(S_{m-1} + T_{m-1} - G_{m-1} - I_{m-1}, r_m + s_m) - c_m$.

If $S_{m-1} - G_{m-1} \geq r_m$, then $g_m = r_m,\ h_m = s_m$, and by Lemma 4.5,

$$\begin{aligned}
&K_m = I_{m-1},\\
&S_{m-1} + T_{m-1} - G_{m-1} - I_{m-1} \geq r_m + T_{m-1} - I_{m-1} = r_m + T_{m-1} - K_m,\\
&S_{m-1} - G_{m-1} + h_m = S_{m-1} - G_{m-1} + s_m \geq r_m + s_m.
\end{aligned}$$

Therefore, $l_m = e_m = min(T_{m-1} - C_m, s_m) + r_m - c_m$.

Case 2. $C_{m-1} + r_m \leq A_{m-1}$.

Then $C_m = C_{m-1} + r_m$, and by Lemma 4.6,

$$\begin{aligned}
&S_{m-1} - G_{m-1} \geq r_m,\\
&g_m = r_m,\\
&h_m = s_m,\\
&S_{m-1} + T_{m-1} - G_{m-1} - I_{m-1} \geq r_m + T_{m-1} - I_{m-1},\\
&S_{m-1} - G_{m-1} + h_m \geq r_m + h_m = r_m + s_m,\\
&e_m = min(S_{m-1} + T_{m-1} + C_{m-1} - A_{m-1} - G_{m-1} - I_{m-1}, T_{m-1} - A_{m-1}, s_m) + r_m - c_m,\\
&l_m = min(T_{m-1} - I_{m-1}, s_m) + r_m - k_m.
\end{aligned}$$

By Lemma 4.7, $S_{m-1} + C_{m-1} - G_{m-1} - A_{m-1} \geq 0$.

If $S_{m-1} + C_{m-1} - G_{m-1} - A_{m-1} = 0$, by Lemma 4.4,

$$\begin{aligned}
e_m &= min(T_{m-1} - I_{m-1}, T_{m-1} - A_{m-1}, s_m) + r_m - c_m\\
&= min(T_{m-1} - I_{m-1}, s_m) + r_m - k_m\\
&= l_m.
\end{aligned}$$

If $S_{m-1} + C_{m-1} - G_{m-1} - A_{m-1} > 0$, by Lemma 4.8,

$$\begin{aligned}
&A_{m-1} = I_{m-1},\\
&S_{m-1} + T_{m-1} + C_{m-1} - A_{m-1} - G_{m-1} - I_{m-1} > T_{m-1} - A_{m-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
e_m &= min(T_{m-1} - A_{m-1}, s_m) + r_m - c_m\\
&= min(T_{m-1} - I_{m-1}, s_m) + r_m - k_m\\
&= l_m.
\end{aligned}$$

□

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